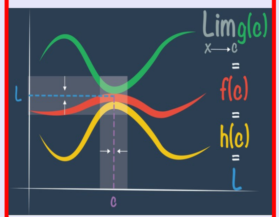
Calculus I

Lecture 11

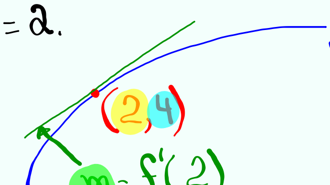


Feb 19-8:47 AM

Class QZ 8

$\boxed{\text{Box Your Final Ans.}}$

Find eqn of the tangent line to the graph of $f(x) = \dfrac{x+2}{x-1}$ at $x = 2$.



$f(x) = \dfrac{x+2}{x-1}$ $\quad f'(x) = \dfrac{1(x-1)-(x+2)\cdot 1}{(x-1)^2}$

$f'(x) = \dfrac{\cancel{x}-1-\cancel{x}-2}{(x-1)^2} = \boxed{\dfrac{-3}{(x-1)^2}}$

$m = f'(2)$

$= \dfrac{-3}{(2-1)^2} = \dfrac{-3}{1^2} = \boxed{-3}$

$y - y_1 = m(x - x_1)$

$y - 4 = -3(x-2)$

$y = -3x + 6 + 4$

$\boxed{y = -3x + 10}$

Jul 2-7:15 AM

Find $f'(x)$

1) $f(x) = (x^2 - 4x)^5$

$f'(x) = 5(x^2-4x)^4 \cdot (2x-4)$

$= 5(x^2-4x)^4 \cdot 2(x-2)$

$= \boxed{10(x^2-4x)^4(x-2)}$

2) $f(x) = \left(\dfrac{x^2-4}{x}\right)^3$

$$\boxed{f'(x) = 3\left(\frac{x^2-4}{x}\right)^2 \cdot \left(1 + \frac{4}{x^2}\right)}$$

$\dfrac{d}{dx}\left[\dfrac{x^2-4}{x}\right] =$

$\dfrac{d}{dx}\left[\dfrac{x^2}{x} - \dfrac{4}{x}\right] =$

$\dfrac{d}{dx}\left[x - 4x^{-1}\right] =$

$1 - 4(-1)x^{-2} =$

$\boxed{1 + \dfrac{4}{x^2}}$

Jul 3-8:17 AM

3) $f(x) = \sqrt{x^3-5x+10}$ $\quad f(x) = (x^3 - 5x + 10)^{1/2}$

$f'(x) = \frac{1}{2}(x^3-5x+10)^{1/2-1} \cdot (3x^2-5)$

$= \frac{1}{2}(x^3-5x+10)^{-1/2}(3x^2-5) = \dfrac{3x^2-5}{2(x^3-5x+10)^{1/2}}$

$= \boxed{\dfrac{3x^2-5}{2\sqrt{x^3-5x+10}}}$

4) $f(x) = \tan(x^2-3x+1)^2$

$f'(x) = \sec^2(x^2-3x+1)^2 \cdot 2(x^2-3x+1)^1 \cdot (2x-3)$

$$\boxed{f'(x) = 2(x^2-3x+1)(2x-3)\sec^2(x^2-3x+1)^2}$$

Jul 3-8:27 AM

5) $f(x) = \sin^4\sqrt{x}$

$f(x) = \left[\sin\sqrt{x}\right]^4$

$f'(x) = \cancel{4}^{2}\left[\sin\sqrt{x}\right]^3 \cdot \cos\sqrt{x} \cdot \dfrac{1}{\cancel{2}\sqrt{x}}$

$$\boxed{f'(x) = \frac{2}{\sqrt{x}}\sin^3\sqrt{x} \cdot \cos\sqrt{x}}$$

6) $f(x) = \cos^3\left(\dfrac{x+2}{x-1}\right) = \left[\cos\left(\dfrac{x+2}{x-1}\right)\right]^3$

$f'(x) = 3\left[\cos\left(\dfrac{x+2}{x-1}\right)\right]^2 \cdot -\sin\left(\dfrac{x+2}{x-1}\right) \cdot \dfrac{-3}{(x-1)^2}$

$$= \boxed{\frac{9}{(x-1)^2}\cos^2\left(\frac{x+2}{x-1}\right)\sin\left(\frac{x+2}{x-1}\right)}$$

Jul 3-8:38 AM

7) $f(x) = \sin^2 x^3 + \cos^2 x^3$

$f(x) = 1$ $\quad \boxed{f'(x) = 0}$

Recall

$\sin^2 A + \cos^2 A = 1$

8) $f(x) = \cos^2\sqrt{x} - \sin^2\sqrt{x}$

Recall

$\cos^2 A - \sin^2 A = \cos 2A$

$f(x) = \cos(2\sqrt{x})$

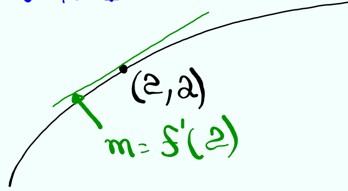
$f'(x) = -\sin(2\sqrt{x}) \cdot \cancel{2} \cdot \dfrac{1}{\cancel{2}\sqrt{x}}$

$$\boxed{f'(x) = \frac{-\sin(2\sqrt{x})}{\sqrt{x}}}$$

Jul 3-8:47 AM

Find eqn of the tan. line to the graph of

$f(x)=\sqrt{\frac{x+2}{x-1}}$ at $x=2$.



$$f(x)=\sqrt{\frac{x+2}{x-1}}$$

$$f(x)=\left(\frac{x+2}{x-1}\right)^{1/2}$$

$$f'(x)=\frac{1}{2}\left(\frac{x+2}{x-1}\right)^{-1/2}\cdot\frac{-3}{(x-1)^2}$$

$$f'(x)=\frac{1}{2}\left(\frac{x-1}{x+2}\right)^{1/2}\cdot\frac{-3}{(x-1)^2}$$

$$f'(x)=\frac{1}{2}\sqrt{\frac{x-1}{x+2}}\cdot\frac{-3}{(x-1)^2}$$

$$m=\frac{1}{2}\sqrt{\frac{2-1}{2+2}}\cdot\frac{-3}{(2-1)^2}$$

$$m=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{-3}{1}=\frac{-3}{4}$$

$$y-y_1=m(x-x_1)$$

$$y-2=\frac{-3}{4}(x-2)$$

$$y=\frac{-3}{4}x+\frac{6}{4}+2$$

$$y=\frac{-3}{4}x+\underbrace{\frac{3}{2}+2}_{\frac{7}{2}}$$

$$\boxed{y=\frac{-3}{4}x+\frac{7}{2}}$$

Jul 3-8:56 AM

Find all points on the graph of $f(x)=(x^2-2x-3)^4$

has horizontal tangent lines.

$f'(x)=4(x^2-2x-3)^3(2x-2)$

$m=0$

$f'(x)=0$

Solve $f'(x)=0$

$$4(x^2-2x-3)^3(2x-2)=0$$

$$4\left[(x+1)(x-3)\right]^3\cdot 2(x-1)=0$$

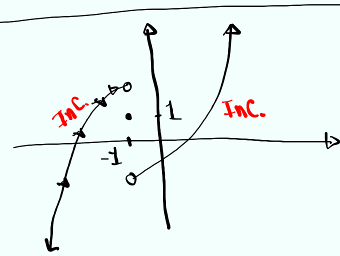
$$8(x+1)^3(x-3)^3(x-1)=0$$

$x+1=0$, $x=-1$

$x-3=0$, $x=3$

$x-1=0$, $x=1$

$(-1,f(-1))$, $(3,f(3))$, $(1,f(1))$



Jul 3-9:07 AM

$f(x) = \sqrt{x}$ $\qquad$ $\boxed{f'(x) = \dfrac{1}{2\sqrt{x}}}$

New method: → Implicit Differentiation

$f(x) = \sqrt{x}$

Square both sides

$[f(x)]^2 = (\sqrt{x})^2$ $\qquad$ $[f(x)]^2 = x$

Now take derivative of both sides

$\frac{d}{dx}\left[(f(x))^2\right] = \frac{d}{dx}[x]$

Chain Rule

$2[f(x)]^1 \cdot f'(x) = 1$

$2f(x)f'(x) = 1$

$f'(x) = \dfrac{1}{2f(x)}$

$\boxed{f'(x) = \dfrac{1}{2\sqrt{x}}}$

Jul 3-9:39 AM

$x^3 - y^3 = 5$ $\qquad$ Find $\dfrac{dy}{dx} = y'$

$\frac{d}{dx}\left[x^3 - y^3\right] = \frac{d}{dx}[5]$

$\frac{d}{dx}[x^3] - \frac{d}{dx}[y^3] = 0$

$3x^2 - 3y^2 \cdot y' = 0$

Isolate y'

$-3y^2 y' = -3x^2$

$y' = \dfrac{\cancel{-3}x^2}{\cancel{-3}y^2}$

$\boxed{y' = \dfrac{dy}{dx} = \dfrac{x^2}{y^2}}$

Jul 3-9:44 AM

$x^2 + y^2 = xy$ Find $\frac{dy}{dx}$

$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[xy]$

$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[x]y + x\frac{d}{dx}[y]$

$2x + 2y\frac{dy}{dx} = 1 \cdot y + x\frac{dy}{dx}$

$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$

$(2y - x)\frac{dy}{dx} = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{2y-x}$

Jul 3-9:49 AM

$\sin y = x$ Find $\frac{dy}{dx}$

$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$

Chain Rule
Implicit Diff.

$\cos y \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \sec y$

$\tan(xy) = y^2$ Find $\frac{dy}{dx}$.

$\frac{d}{dx}[\tan(xy)] = \frac{d}{dx}[y^2]$

$\sec^2(xy) \cdot \frac{d}{dx}[xy] = 2y\frac{dy}{dx}$

$\sec^2(xy) \cdot \left[1 \cdot y + x \cdot \frac{dy}{dx}\right] = 2y\frac{dy}{dx}$

$y\sec^2(xy) + x\sec^2(xy)\frac{dy}{dx} = 2y\frac{dy}{dx}$

$y\sec^2(xy) = 2y\frac{dy}{dx} - x\sec^2(xy)\frac{dy}{dx}$

$y\sec^2 xy = (2y - x\sec^2(xy))\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y\sec^2(xy)}{2y - x\sec^2(xy)}$

Jul 3-9:55 AM

Find $\frac{dy}{dx}$ for $x\sin y + y\sin x = 1$

$$\frac{d}{dx}\left[x\sin y + y\sin x\right] = \frac{d}{dx}[1]$$

$$\frac{d}{dx}\left[x\sin y\right] + \frac{d}{dx}\left[y\sin x\right] = 0$$

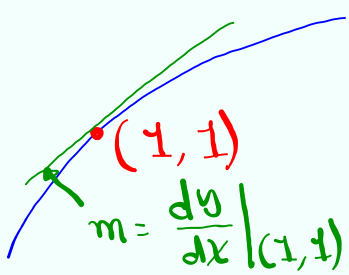
$$1\cdot\sin y + x\cdot\cos y\cdot y' + y'\cdot\sin x + y\cdot\cos x = 0$$

$$(x\cos y + \sin x)\,y' = -\sin y - y\cos x$$

$$\boxed{y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}}$$

Jul 3-10:10 AM

Find equation of the tan. line to the graph $x^2 + xy + y^2 = 3$ at $(1,1)$.



$$x^2 + xy + y^2 = 3$$

$$2x + y + xy' + 2yy' = 0$$

Plug in $(1, 1)$

Replace y' with m

$$2(1) + 1 + 1m + 2m = 0$$

$$3 + 3m = 0$$

$$3m = -3$$

$$\boxed{m = -1}$$

$$y - y_1 = m(x - x_1)$$

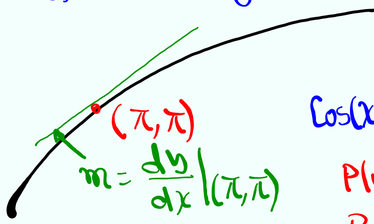
$$y - 1 = -1(x - 1)$$

$$\boxed{y = -x + 2}$$

Jul 3-10:19 AM

Find eqn of tan. line to the graph of

$\sin(x+y) = 2x - 2y$ at (π, π).



$\sin(x+y) = 2x - 2y$

$\cos(x+y)\cdot[1+y'] = 2 - 2y'$

Plug in (π,π),
Replace y' with m

$\underbrace{\cos 2\pi}_{1}(1+m) = 2 - 2m$

$1 + m = 2 - 2m$

$m + 2m = 2 - 1$

$3m = 1$

$\boxed{m = \frac{1}{3}}$

$y - y_1 = m(x - x_1)$

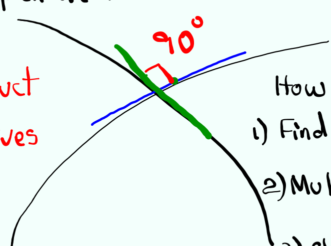
$y - \pi = \frac{1}{3}(x - \pi)$

$y = \frac{1}{3}x - \frac{\pi}{3} + \pi$

$\boxed{y = \frac{1}{3}x + \frac{2\pi}{3}}$

Jul 3-10:31 AM

Two curves are orthogonal if their tangent lines at their intersection points are perpendicular to each other.



Show product of derivatives is -1.

How to verify
1) Find derivatives
2) Multiply derivatives
3) Show is -1.

$x^2 + y^2 = r^2$
Circle

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = \frac{-2x}{2y}$

$y' = \frac{-x}{y}$

$ax + by = 0$
line $\longrightarrow by = -ax$

$a + by' = 0$

$by' = -a$

$y' = \frac{-a}{b}$

$\frac{-x}{y} \cdot \frac{-a}{b} = \frac{ax}{by} = \frac{\cancel{ax}}{-\cancel{ax}} = -1$

Jul 3-10:40 AM

Show $\boxed{x^2+y^2=ax}$ and $\boxed{x^2+y^2=by}$ are orthogonal to each other.

$2x + 2yy' = a$

$y' = \dfrac{a-2x}{2y}$

$2x + 2yy' = by'$

$2x = by' - 2yy'$

$2x = (b-2y)y'$

$y' = \dfrac{2x}{b-2y}$

$\dfrac{a-2x}{\cancel{2}y} \cdot \dfrac{\cancel{2}x}{b-2y} =$

$\dfrac{x(a-2x)}{y(b-2y)} = \dfrac{ax-2x^2}{by-2y^2} = \dfrac{x^2+y^2-2x^2}{x^2+y^2-2y^2}$

$= \dfrac{y^2-x^2}{x^2-y^2} = \boxed{-1}$

Recall from Basic Math

$\dfrac{a-b}{a-b} = 1, \quad \dfrac{a-b}{b-a} = -1$

Jul 3-10:49 AM

Estimate $\sqrt[3]{65}$

1) from Calculator $\to$ $\sqrt[3]{65} \approx 4.021$ ✓

Common Sense $\sqrt[3]{65} \approx \sqrt[3]{64} = 4$

Linear Approximation

$f(x) = \sqrt[3]{x}$

$a = 64$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1}$

$= \frac{1}{3}x^{-\frac{2}{3}}$

$= \dfrac{1}{3\sqrt[3]{x^2}}$

$f(x) \approx f(a) + f'(a)(x-a)$

$\sqrt[3]{x} \approx \sqrt[3]{64} + f'(64)(x-64)$

$f'(64) = \dfrac{1}{3\sqrt[3]{64^2}} = \dfrac{1}{3\cdot 16} = \dfrac{1}{48}$

$\sqrt[3]{x} \approx 4 + \frac{1}{48}(x-64)$

$\sqrt[3]{65} \approx 4 + \frac{1}{48}(65-64)$

$= 4 + \frac{1}{48}\cdot 1 = \dfrac{193}{48}$

$\approx 4.020\overline{83}$

$\approx \boxed{4.021}$

Jul 3-10:58 AM